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Exponentially-Fitted Runge-Kutta Nystrom Method of Order Three for Solving Oscillatory Problems

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ABSTRACT

In this paper the exponentially fitted explicit Runge-Kutta Nystrom method is proposed for solving special second-order ordinary differential equations where the solution is oscillatory. The exponentially fitting is based on given Runge-Kutta Nystrom (RKN) method of order three at a cost of three function evaluations per step. Here, we also developed the trigonometrically-fitted RKN method for solving initial value problems with oscillating solutions. The numerical results compared with the existing explicit RKN method of order three which indicates that the exponentially fitted explicit Runge-Kutta Nystrom method is more efficient than the classical RKN method.

Keywords: Exponentially-fitting, Runge-Kutta Nystrom method, oscillating solutions.

1. INTRODUCTION

Consider the second order ordinary differential equation (ODEs)

$$\begin{aligned} y''(x) &= f(x, y(x)), \qquad x \in [x_0 \ X], \\ y(x_0) &= y_0, \qquad y'(x_0) = y'_0 \end{aligned} \tag{1}$$

For approximating solution of problem in equation (E1) many different methods have been studied. One of the most common method for solving numerically (1) is Runge-Kutta Nystrom (RKN) methods (see Dormand (1996)). Exponentially fitting is a procedure for an efficient of exponential, trigonometric or hyperbolic function for solving ODEs whose solutions are oscillatory. Simos (2002) and Franco (2004) proposed the different approaches for exponentially fitted on explicit Runge-Kutta Nystrom methods. In this paper we propose the exponentially and trigonometrically fitting for third order, 3-stages Runge-Kutta Nystrom method with minimum phase-lag derived by Norazak *et al.* (2010) also the exponentially fitting for RKN method of order three with normal set of coefficients. In section 2, for exponentially fitting the RKN method integrates exactly the functions $\exp(\pm iwx)$ where $w \in R$. In Section 3, the construction of trigonometrically fitting RKN methods by integrating exactly the functions $\exp(\pm iwx)$ where $w \in R$ and $i = \sqrt{-1}$, is proposed. The number of test problems are experienced and the numerical results compared with existing RKN method of order 3 derived by Dormand (1996), are given in last section.

The explicit RKN method is given by

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i k_i, \quad y'_{n+1} = y'_n + h \sum_{i=1}^{s} \bar{b}_i k_i,$$

$$k_1 = f(x_n, y_n), \quad k_1 = f(x_n + c_i h, y_n + hc_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j), i = 2, \ldots, s. \qquad (2)$$

where $c = [c_1, c_2, \cdots, c_s]^T$, $b = [b_1, b_2, \cdots, b_s]^T$, $\bar{b} = [\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_s]^T$, and $(s \times s)$ matrix $A = [a_{ij}]$. The characteristic equation for RKN formulas (2) is given by Franco (2004),

$$\zeta^2 + tr(D(z^2))\zeta + \det(D(z^2)) = 0$$

where $D = \begin{bmatrix} A(z^2) & B(z^2) \\ A'(z^2) & B'(z^2) \end{bmatrix}$ and A, A', B, B' are polynomial in z^2. For RKN method with characteristic equation

$$\phi(z) = z - \cos^{-1}\left(\frac{R(z^2)}{2\sqrt{S(z^2)}}\right), \quad \alpha(z) = 1 - \sqrt{S(z^2)},$$

where $R(z^2) = tr(D)$ and $S(z^2) = \det(D)$, $\phi(z)$ is *phase-lag* and $\alpha(z)$ is *dissipation*. RKN method has *phase-lag* order q when $\phi(z) = O(z^{q+1})$ and RKN method has *dissipation* order when $\alpha(z) = O(z^{r+1})$.

Norazak *et al.* (2010) developed the explicit RKN method of algebraic order $p=3$ and phase-lag order $q=6$ at a cost of three function evaluations per step of integration. Here, we chose the values of free parameters from his method.

2. EXPONENTIALLY-FITTED THIRD ORDER RKN METHOD

To construct the exponentially-fitted RKN method, we require that all stages of third order RKN method integrate exactly to the function $\exp(\pm wx)$. We have

$$\begin{aligned}
&e^{c_2 v}-1-c_2 v-v^2 a_{21}=0,\ \ e^{-c_2 v}-1+c_2 v-v^2 a_{21}=0,\\
&e^{c_3 v}-1-c_3 v-v^2(a_{31}+a_{32}e^{c_2 v})=0,\ \ e^{-c_3 v}-1+c_3 v-v^2(a_{31}+a_{32}e^{-c_2 v})=0,\\
&e^{v}-1-v-v^2(b_1+b_2 e^{c_2 v}+b_3 e^{c_3 v})=0,\\
&e^{-v}-1+v-v^2(b_1+b_2 e^{-c_2 v}+b_3 e^{-c_3 v})=0,\\
&e^{v}-1-v(\bar{b}_1+\bar{b}_2 e^{c_2 v}+\bar{b}_3 e^{c_3 v})=0,\ \ e^{-v}-1+v(\bar{b}_1+\bar{b}_2 e^{-c_2 v}+\bar{b}_3 e^{-c_3 v})=0,
\end{aligned}$$

where $v=wh, w\in R$. Using relations:

$$\cosh(v)=\frac{e^{v}+e^{-v}}{2},\ \ \sinh(v)=\frac{e^{v}-e^{-v}}{2}.$$

the following order conditions are obtained:

$$\begin{gathered}
\cosh(c_2 v)=1+v^2 a_{21},\ \ \cosh(c_3 v)=1+v^2(a_{31}+a_{32}\cosh(c_2 v)),\\
\sinh(c_3 v)=c_3 v+v^2 a_{32}\sinh(c_2 v)),\\
\cosh(v)=1+v^2(b_1+b_2\cosh(c_2 v)+b_3\cosh(c_3 v)),\\
\sinh(v)=v+v^2(b_2\sinh(c_2 v)+b_3\sinh(c_3 v)),\\
\cosh(v)=1+v(\bar{b}_2\sinh(c_2 v)+\bar{b}_3\sinh(c_3 v)),\\
\sinh(v)=v(\bar{b}_1+\bar{b}_2\cosh(c_2 v)+\bar{b}_3\cosh(c_3 v)).
\end{gathered}$$

We set the values of parameters $c_2=\frac{1}{2}, c_3=1$ and $a_{31}=\frac{1}{6}$ from RKN methods with minimum phase-lag derived by Norazak *et al.* (2010), into above order conditions, so we have (EF-N method):

$$a_{21} = 0.127625965,\ a_{32} = 0.3338110152,$$
$$b_1 = 0.1646217452,\ b_2 = 0.3347099233,$$
$$b_3 = 0.0006683314237,\ \bar{b}_1 = 0.1652900767,\ \bar{b}_2 = 0.6694198461,\ \bar{b}_3 = \bar{b}_1.$$

Also by choosing $c_2 = \frac{1}{2}, c_3 = 1$ and $a_{31} = 0$ as free parameters, we have (EF method):

$$a_{21} = 0.127625965,\ a_{32} = 0.4816141626,$$
$$b_1 = 0.1646217452,\ b_2 = 0.3347099233,$$
$$b_3 = 0.0006683314237,\ \bar{b}_1 = 0.1652900767,\ \bar{b}_2 = 0.6694198461,\ \bar{b}_3 = \bar{b}_1.$$

3. TRIGNOMETRICALLY-FITTED THIRD ORDER RKN METHOD

To construct the Trigonometrically-fitted RKN method, we require that all stages of third order RKN method integrate exactly to the function $\exp(\pm iwx)$ where $i = \sqrt{-1}$, so we have

$$e^{ic_2v} - 1 - ic_2v + v^2 a_{21} = 0,\ \ e^{-ic_2v} - 1 + ic_2v + v^2 a_{21} = 0,$$
$$e^{ic_3v} - 1 - ic_3v + v^2(a_{31} + a_{32}e^{ic_2v}) = 0,\ e^{-ic_3v} - 1 + ic_3v + v^2(a_{31} + a_{32}e^{-ic_2v}) = 0,$$
$$e^{-ic_3v} - 1 + ic_3v + v^2(a_{31} + a_{32}e^{-ic_2v}) = 0,$$
$$e^{iv} - 1 - iv + v^2(b_1 + b_2e^{ic_2v} + b_3e^{ic_3v}) = 0,$$
$$e^{-iv} - 1 + iv + v^2(b_1 + b_2e^{-ic_2v} + b_3e^{-ic_3v}) = 0,\ \ ie^{iv} - i + v(\bar{b}_1 + \bar{b}_2e^{ic_2v} + \bar{b}_3e^{ic_3v}) = 0,$$
$$ie^{-iv} - i - v(\bar{b}_1 + \bar{b}_2e^{-ic_2v} + \bar{b}_3e^{-ic_3v}) = 0,$$

where $v = wh, w \in R.$ Using $\cos(v) = \dfrac{e^{iv} + e^{-iv}}{2},\ \sin(v) = \dfrac{e^{iv} - e^{-iv}}{2i},$ the following order conditions are obtained

$$\cos(c_2v) = 1 - v^2 a_{21},\ \ \cos(c_3v) = 1 - v^2(a_{31} + a_{32}\cos(c_2v)),$$
$$\sin(c_3v) = c_3v - v^2 a_{32}\sin(c_2v),\ \ \cos(v) = 1 - v^2(b_1 + b_2\cos(c_2v) + b_3\cos(c_3v)),$$
$$\sin(v) = v - v^2(b_2\sin(c_2v) + b_3\sin(c_3v)),\ \ \cos(v) = 1 - v(\bar{b}_2\sin(c_2v) + \bar{b}_3\sin(c_3v)),$$
$$\sin(v) = v(\bar{b}_1 + \bar{b}_2\cos(c_2v) + \bar{b}_3\cos(c_3v)).$$

We set the values of parameters $c_2 = \frac{1}{2}, c_3 = 1$ and $a_{31} = \frac{1}{6}$ from RKN methods with minimum phase-lag derived by Norazak *et al.* (2010) into above order coditions, so we have (TF-N Method):

$$a_{21} = 0.1224174381,\ a_{32} = 0.3339070764,$$
$$b_1 = 0.1687901678,\ b_2 = 0.3319319376,\ b_3 = -0.0007221071160,$$
$$\bar{b}_1 = 0.1680680599,\ \bar{b}_2 = 0.6638638777,\ \bar{b}_3 = \bar{b}_1.$$

4. NUMERICAL RESULTS

To illustrate the efficiency of new methods the following problems are solved for $x \in [0,50]$ and $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$ and numerical results are compared with existing third order RKN method derived by Dormand (1996). In addition, the following abbreviations are used in Figures 1-11.

- EF-N: exponentially-fitted Runge-Kutta Nystrom method of order three with minimum phase-lag.
- EF: exponentially-fitted Runge-Kutta Nystrom method of order three when $a_{31} = 0$.
- TF-N: trigonometrically-fitted Runge-Kutta Nystrom method of order three with minimum phase-lag.
- RKND3: classical Runge-Kutta Nystrom method of order three derived by Dormand (1996).

Problem 1. (Norazak *et al* (2010)).

$$y'' = -y + 0.001\cos(x), \quad y(0) = 1, \quad y'(0) = 0.$$

Exact solution: $y(x) = \cos(x) + 0.0005x\sin(x).$

Problem 2. (Franco (2006)).

$$y_1'' + y_1 = 0.001\cos(x), \quad y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2'' + y_2 = 0.001\sin(x), \quad y_2(0) = 0, \quad y_2'(0) = 0.9995.$$

Exact solutions:

$$y_1(x) = \cos(x) + 0.0005x\sin(x), \;\; y_2(x) = \sin(x) - 0.0005x\cos(x).$$

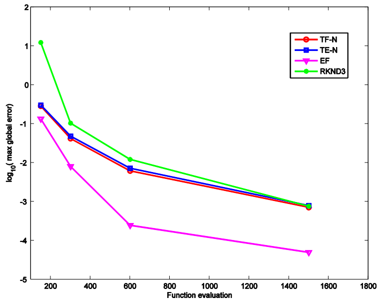

Figure 1: Logarithm of maximum global error versusnumber of function evaluations for solving problem 1

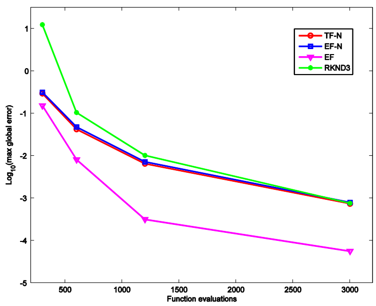

Figure 2: Logarithm of maximum global error versus number of function evaluations for solving problem 2

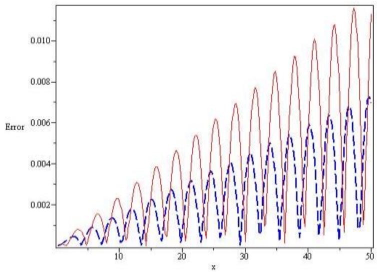

Figure 3: Maximum global error of EF-N method (dash line) and RKND3 (solid line) in solving problem 1

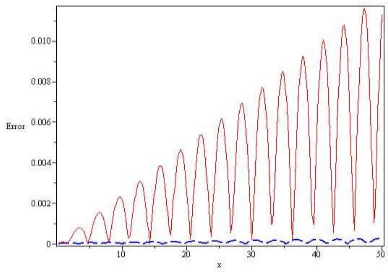

Figure 4: Maximum global error of EF method (dash line) and RKND3 (solid line) in solving problem 1

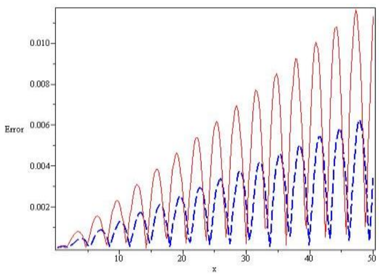

Figure 5: Maximum global error of F-N method (dashline) and RKND3 (solid line) in solving problem

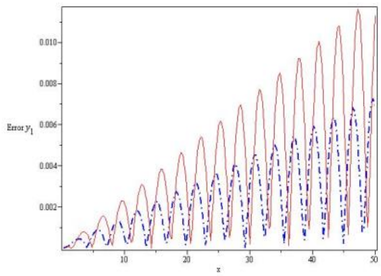

Figure 6: Maximum global error of EF-N method (dash line) and RKND3 (solid line) for $y1$ in solving problem 2

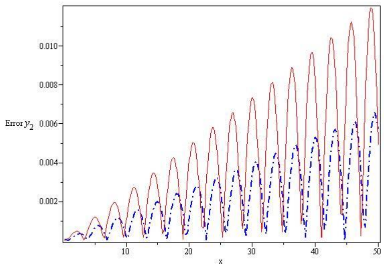

Figure 7: Maximum global error of EF-N method (dash line) and RKND3 (solid line) for *y*2 in solving problem 2

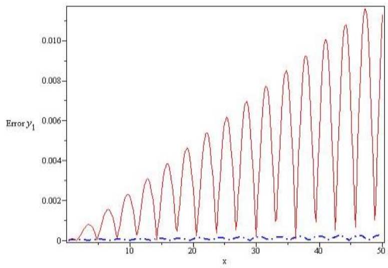

Figure 8: Maximum global error of EF method (dash line) and RKND3 (solid line) for *y*1 in solving problem 2

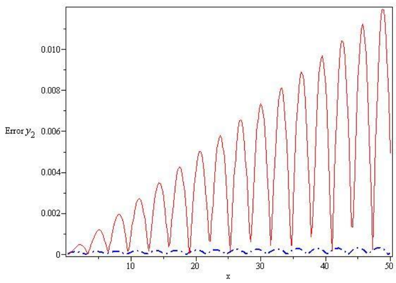

Figure 9: Maximum global error of EF method (dash line) and RKND3 (solid line) for *y*2 in solving problem 2

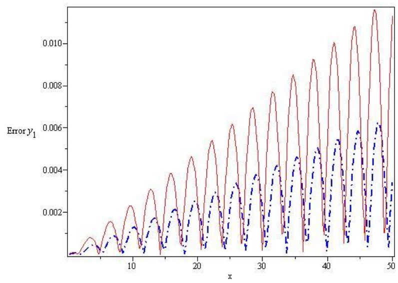

Figure 10: Maximum global error of TF-N method (dash line) and RKND3 (solid line) for *y*1 in solving problem 2

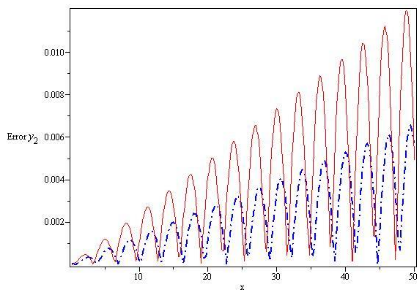

Figure 11: Maximum global error of TF-N method (dash line) and RKND3 (solid line) for *y*2 in solving problem 2

5. CONCLUSION

In this study we constructed the exponentially- fitted and trigonometrically- fitted Runge-Kutta Nystrom method of order three with 3-stages. The new methods give the accurate results for problems with oscillation solutions while the step size *h* was not chosen as small value. Numerical examples indicate that the exponentially fitted RKN methods are more efficient than given classical RKN method.

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